



GEOMETRIC BROWNIAN MOTION: AN ALTERNATIVE TO HIGH-FREQUENCY TRADING FOR SMALL INVESTORS

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ABSTRACT

High-frequency trading (HFT) involves short-term, high-volume market operations to capture profits. To a large extent, these operations take advantage of early access to information using fast and sophisticated technological tools running on supercomputers. However, high-frequency trading is inaccessible to small investors because of its high cost. For this reason, price prediction models can substitute high-frequency trading in order to anticipate stock market movements. This study is the first to analyze the possibility of applying Geometric Brownian Motion (GBM) to forecast prices in intraday trading of stocks negotiated on two different stock markets: (i) the Brazilian stock market (B3), considered as a low liquidity market and (ii) the American stock market (NYSE), a high liquidity market. This work proposed an accessible framework that can be used for small investors. The portfolios formed by Geometric Brownian Motion were tested using a traditional risk measure (mean-variance). The hypothesis tests showed evidences of promising results for financial management.

Keywords: Geometric brownian motion, High-frequency trading, Algorithmic trading, Financial engineering, Statistical inference



1. INTRODUCTION

In financial markets, decision-making involves four main variables: profitability, risk, liquidity, and income taxation. In this context, an alternative strategy used by many investors includes buying and selling stocks on the same day, known as intraday trading. These applications allow satisfactory results from trading high-liquidity stocks on the same day. The short interval between buying and selling allows higher assertiveness in making inferences about their risk, which depends primarily on random processes (COLMAN; WIENANDTS; DE PIETRO, 2013).

In addition to intraday trading, negotiations in major stock markets have been drastically improved in recent years by using advanced technological resources such as algorithmic trading (AT). Hendershott et al. (2011) have shown that AT is usually defined as the use of computational algorithms to make specific business decisions, send orders, and manage these orders after order submission.

High-frequency trading (HFT) can be considered as AT. The Securities and Exchange Commission (SEC, 2014) declared that, although HFT currently represents approximately 50% of the volume traded in the United States, the concept of HFT is still unclear. However, according to Menkveld (2013), the characteristics of users of this type of trading including (i) predominance of zero inventory positions at the end of each day; (ii) frequency of trading at time intervals $<5s$; (iii) profit obtained primarily by transaction spreads (sale price minus purchase price); (iv) use of passive strategies in most cases in line with market price opportunities; (v) thousands of transactions a day on average; (vi) negotiations of orders including large batches of stocks; and (vii) operation in markets with advantageous operating taxes and technological resources compatible with market needs.

The response speed is one of the key characteristics of this category. The use of algorithms incorporated into powerful supercomputers allows profits in operations executed in stock market trading and completing these operations in fractions of seconds.

Liquidity is an essential attribute for trading intervals of ultra-high frequency ($<5s$), high frequency (<1 min), or medium frequency (<1 h) in intraday operations (MENKVELD, 2013). In this study, the traditional concept of liquidity was used, i.e., speed and ease with which stock can be converted into cash.

The analysis of HFT characteristics raises the question of whether it is possible for small investors (private individuals and small and medium-sized enterprises) to apply HFT concepts to conventional computing resources.

One obstacle found in this study was the lack of studies on intraday trading in the Brazilian stock market. In this sense, this study offers an additional contribution showing whether this kind of transaction is feasible in Brazil.

In recent years, the use of AT has become common in major financial markets worldwide (NYSE, CME-Chicago, NASDAQ, Euronext, Chi-X, B3). AT was first used in the United States capital market in 1990 (CHABOUD; CHIQUOINE; HJALMARSSON; VEGA, 2014). However, there is still controversy over the number of resources available to professional and small investors. Pentagna (2015) has found that HFT firms take advantage of short time intervals to outperform traditional investors and earn a slightly higher profit margin.

Geometric Brownian Motion (GBM) is a Markovian process, in which future prices are predicted by considering the last observed record (LAGE, 2011). Developing a stochastic model that efficiently represents the trading prices of assets (stocks, oil, soy, coffee, steel, rubber) is an advantageous feature for small investors, whose access to computing resources from high-frequency trading is limited. In addition, mathematical models that can form lower risk portfolios mean lower chances of systemic crises for society as a whole.

The objectives of this study were (i) to analyze the feasibility of applying GBM to forecast prices in intraday trading (during market hours and its variations) by small investors as an alternative to the high-frequency trading adopted by large corporations. Data with a time interval of 30 min and 60 min were collected from the Bloomberg system from September 2014 to April 2015, with free registrations of market operations made at 1-min intervals in the Brazilian Stock Exchange (B3) and New York Stock Exchange (NYSE); and (ii) to evaluate the profitability of investment portfolios created by GBM-based price forecasts using the mean-variance (MV) optimization model of Markowitz.

This study presents the (i) fundamentals of the GBM and MV models; (ii) methodology used; (iii) characterization of the samples and applied tests; (iv) results; (v) discussion and (vi) conclusions.

2. THEORETICAL BASIS

In this section, we present the main references used for evaluating the theoretical basis of the study using the GBM and MV models for portfolio optimization.



2.1. Geometric Brownian Motion

GBM is a stochastic model discovered by Robert Brown in 1827 by observing the continuous movement and irregular trajectories of pollen grains in an aqueous suspension. The stochastic process that describes GBM properties was first defined by Wiener (1923). Ito (1944) developed the fundamentals of stochastic calculus to allow the differential calculation of Brownian stochastic processes. Over time, GBM was applied to several types of situations (BODINEAU; GALLAGHER; SAINT-RAYMOND, 2016; SEYF; NIKAAEIN, 2012; ZHANG; ZHOU, 2015). The famous model developed by Black and Scholes (1973), Nobel Prize in Economics, adopted GBM for forecasting market stock prices (IWAKI; LUO, 2013; KOGAN; PAPANIKOLAOU, 2014).

GBM studies focus on market indices to assess the efficiency of forecasting market stock prices in short intervals (ABIDIN; JAFFAR, 2012; REBOREDO, RIVERA-CASTRO; MIRANDA; GARCIA-RUBIO, 2013; REDDY; CLINTON, 2016; ZHOU, 2015). However, few studies applied GBM based prediction to form portfolios in different stock markets and in HFT conditions, especially in the Brazilian stock market. A summary of studies on the performance of GBM in different scenarios is shown in Table 1.

Table 1: GBM applications at different time intervals.

Reference	Description	Main results
Abdin and Jaffar (2012)	Evaluated the performance of GBM for predicting daily closing prices of 77 small- and medium-sized enterprises in the stock market of Malaysia over a 30-day period.	GBM presented a high forecast performance for up to 14 days.
Reboredo et al. (2013)	Investigated the period necessary for prices to adjust to GBM by analyzing stock prices at 1-min intervals on two stock market indices, one exchange market, and one Spanish stock during 74 trading days.	There was a quick adjustment to GBM for time intervals of <1 day. GBM presented good performance for forecasting stock prices.
Reddy and Clinton (2016)	Tested the efficiency of GBM to predict stock prices at daily closing prices of 50 large Australian companies in the year 2013.	In the evaluated period, GBM showed that actual prices were like those of projected prices in more than 50% of the cases.
Zhou (2015)	Tested the efficiency of GBM for forecasting daily closing prices of one option during 81 consecutive trading sessions from January to May 2014.	GBM presented favorable results.

GBM is also a Markovian stochastic process in which only the last observed record is considered for forecasting future prices. Hillier and Lieberman (2015) have reported that a Markovian stochastic process is defined as an indexed set of random variables $\{X_t\}$, where the

index t includes a given set T . In most cases, it is assumed that T is the set of nonnegative integers and X_t represents a measurable characteristic of interest at time t .

In the average GBM, the most common equations used to generate a stochastic process of a random variable S (price) assuming an initial value s_0 in t_0 and a final value in t_f are (ROSS, 2014; SIGMAN, 2006):

$$S(t_f) = s_0 e^{\left[\left(\mu + \frac{\sigma^2}{2} \right) (t_f - t_0) \right]} \quad (1)$$

$$\text{var} [S(t)] = s_0^2 e^{\left[(2\mu + \sigma^2)(t_f - t_0) \right]} \left[e^{\sigma^2(t_f - t_0)} - 1 \right] \quad (2)$$

2.2. Mean-Variance Optimization Model

Markowitz (1952) proposed a portfolio optimization model known as the mean-variance (MV) model, which is based on the risk-return duality and defines the optimum combination of stocks at the lowest possible risk to surpass a rate of return. This model formed the basis of modern economic theory and was awarded the Nobel Prize in Economics. The adopted risk measure was variance, which is usually obtained by analyzing historical data of the evaluated stocks.

The mean-variance model created by Markowitz is shown below.

$$\text{Min} Z = \sum_{i=1}^N \sum_{j=1}^N x_i x_j \sigma_{ij} \quad (3)$$

Subject to:

$$\sum_{i=1}^N x_i \mu_i \geq \rho \quad (4)$$

$$\sum_{i=1}^N x_i = 1 \quad (5)$$

$$x_i \geq 0 \quad i = 1, \dots, N \quad (6)$$

Where:

N – Number of stocks evaluated in the portfolio;

x_i – Percentage of capital to be invested in stock i ;

σ_{ij} – Covariance between stocks i and j ;

μ_i – Expected rate of return of stock i ;

ρ – Minimum rate of return defined by the investor.

The objective function (3) of the model is to minimize the risk of the portfolio, and the risk is represented by the covariance between stocks. The first restriction (4) presented in the model is the rate of return expected by the investor, which should be met by the portfolio used. The second restriction (5) requires that all capital must be invested. The latter restriction avoids negative rates of return for any of the stocks.

There are three primary data inputs: (i) expected rates of return of the candidate stocks; (ii) correlation between the rates of return; and (iii) covariance. The model is based on portfolio valuation considering the expected stock price (return) and variance of the rates of return (risk). Therefore, when choosing between two portfolios with the same risk, investors should choose the one with the highest return.

The principle of optimal allocation of available resources based on risk developed by Markowitz is highly applicable to other areas, which amplified the relevance of this study. In particular, the covariance matrix, which represents the dependency relationships between the stocks involved, is used as a risk management strategy in other scientific fields. Some of the main contributions of the model are: (i) portfolio optimization (CASTELLANO; CERQUETI, 2014; CUI; GAO; LI; LI, 2014; LIOUI ; PONCET, 2016; QIN, 2015); (ii) risk conceptualization (MCNEIL; FREY; EMBRECHTS, 2015); (iii) risk measures (AHMADI-JAVID, 2012; CHOI; CHUI, 2012; MARKOWITZ, 2014); and (iv) stochastic calculus (KHARROUBI; LIM; NGOUPEYOU, 2013).

Despite the development of new risk measures (ROCKAFELLAR; URYASEV, 2000; NOYAN; RUDOLF, 2013), the MV model is still a relevant reference for improving portfolio optimization.

3. METHODOLOGY

This study is characterized as applied research and quantitatively evaluated the problem of price projection when creating investment portfolios in intraday trading. The public data (prices) used in the study were collected in a cash market database from the Bloomberg stock market. Bloomberg is one of the leading providers of business market information worldwide. This database was also chosen because it is used in several markets. The data cover trading records made at 1-min intervals from September 18, 2014, to April 2, 2015, on B3 (Brazil) and NYSE (USA). This period comprised 196 calendar days or 131 and 139 trading sessions on B3 and NYSE, respectively.



The trading floor data from the Bloomberg system of B3 and NYSE comprised a population of 445 and 3255 traded stocks, respectively, beginning and ending at 9 a.m. and 4 p.m., respectively, in the analyzed period. The trading sessions were divided into two-time intervals: (i) 14 intervals of 30 min and seven intervals of 60 min on B3 and (ii) 12 intervals of 30 min and six intervals of 60 min on the NYSE.

3.1. Sample Size and Assessment of Normality of Residuals Obtained in GBM Predictions

The use of GBM for predicting prices implies residues with normal distribution (LAGE, 2011). Therefore, assessing this property in the predictions is essential to guarantee compliance with the basic assumptions.

The statistical test used for this analysis was the non-parametric Anderson-Darling (AD) test, which is available as default in software Action (integrated into Microsoft Excel) and is considered suitable for normality tests (CARRADORI; RAMOS, 2014; SHIN; JUNG; JEONG; HEO, 2012). A level of significance α of 5% was considered in the tests. Therefore, rejection of the null hypothesis occurred in cases in which the *p-value* was less than 0.05.

There are no references from other studies that could be used to estimate the population proportion of interest (success rate of the normality test). Under these conditions, Anderson et al. (2013) recommend a planned p^* equal to 0.50 (50%). The use of p^* equal to 0.50 allows obtaining the largest possible sample size and ensures that the sample size is enough to reach the desired error margin. In fact, the error margin calculated after sample definition should be less than the error margin adopted before.

A 95% confidence level and a 7.5% error margin for the intraday data in both capital markets were assumed to calculate the sample size. Therefore, the recommended sample size for each 60- and 30-min section was 171, totaling 684 tests considering both markets.

The number of trading records in both markets at 1-min intervals in the analyzed period was 1.87×10^8 . The manipulation of these records was unfeasible. All the stocks from the two capital markets were initially classified in descending order of trading volume (liquidity) and were later selected by random sampling.

Based on the estimated sample size, we chose 30 shares from B3 and 45 shares from the NYSE. In addition, the number of days selected at random for applying the normality tests on B3 and NYSE was 30 and 45 days, respectively. This number of stocks met the statistical

requirements necessary to evaluate the time intervals, ensured total reliability in executing the tests, and is consistent with the study objectives regarding accessibility to small investors.

The number of stocks used did not jeopardize the results because the inclusion of more stocks increases the chances of creating portfolios and making profit by incorporating more liquid stocks. The following intervals were tested: (i) 182 intervals of 60-min and 364 intervals of 30-min on B3 and (ii) 190 intervals of 60 min and 269 intervals of 30 min on the NYSE. In total, 1005 samples were distributed as follows: (i) 372 intervals of 60 min and (ii) 633 intervals of 30 min.

The test was applied for the 60- and 30-min samples from both capital markets. Based on the studies by Iman and Conover (1983) and by Sheng et al. (2015), the first 60% of the price records of each selected time interval (60 and 30 min) were used to adjust the distribution (determination of μ and σ) and the remaining 40% of the price records were used for predicting prices. In the case of the intraday market price predictions made by GBM, the results of the hypothesis test generated internally by the software Action were used.

Extracts with less than eight observations necessary for the AD normality test were discarded. The number of successful predictions (adherence to normal distribution) in 60- and 30-min time intervals was counted and divided by the total of valid periods in each interval (e.g., 5 successful intervals / 6 valid intervals = 83.3%). This study analyzed the following hypothesis for each evaluated time interval of each chosen stock/day:

- ***H1: The residuals obtained by predicting prices using GBM follow a normal distribution.***

3.2. Evaluation of the Success Rates of GBM Predictions

The stocks chosen at random were evaluated for six 60-min intervals and twelve 30-min intervals on the NYSE and seven 60-min intervals and fourteen 30-min intervals on the B3. The quoted prices exclude the cases with fewer than eight price observations, the minimum value required for the AD test, or stocks without results in one of the time intervals.

After calculating the rate of success of price prediction using GBM for each selected interval, the time interval (60 or 30 min) that achieved the best result was determined. A *t*-test on the difference in the means of related (dependent) samples in the 60- and 30-min time intervals was conducted using the Microsoft Excel® data analysis functions. The following hypothesis was evaluated:

- **H2: The mean success rate in the normality test in 30-min intervals is higher than that in 60-min intervals.**

3.3. Evaluation of the Profitability of the Portfolios Created using the MV Model

The profitability of the portfolios formed using the MV model was tested as follows:

- 30-min time intervals were selected (according to the result obtained in the previous phase) to assess the history of the stock price;
- For each 30-min interval, the stock price history from the first to the 18th minute was determined (phase of estimation of GBM parameters);
- Portfolios were formed by predicting prices using GBM for the remaining 12 min of each time interval;
- For all time intervals, a single strategy was defined as the assembly (purchase) of the portfolio in the 18th minute and its disassembly (sale) in the 30th minute;
- The prices effectively practiced by the market in the assembly (18 minutes) and disassembly (30 minutes) were recorded to determine the profit or loss of the formed portfolio;
- The Markowitz MV model was used to select the stocks in the portfolios (ABENSUR, 2014).

In this phase of testing, time intervals were chosen at random for applying the MV optimization model. The independent effect of stock prices was ensured by using intervals of at least 2 days between the trading sessions, and Mondays and Fridays were avoided whenever possible because these days were used by market managers to adjust their investment allocation strategies (KEIM; STAMBAUGH, 1984). A total of 88 trading sessions were selected for testing the formation of the portfolios.

Optimization simulations were made using the Solver optimization application available in Microsoft Excel®. The results of the minimum-risk portfolios formed were subsequently subjected to a *Z-test* of the mean, according to the following hypothesis:

- **H3: The mean rate of return was positive (profit).**

4. RESULTS

This section includes (i) evaluation of normality of the residuals and comparison of price forecasts; and (ii) evaluation of the profitability of the investment portfolios formed by GBM forecasts. A summary of the statistical treatments applied in this study is presented.

4.1. Evaluation of the Normality of Residuals and Comparison of Price Predictions

The AD tests were executed on the GBM-based price projections obtained from each of the 372 60-min time intervals and 633 30-min time intervals evaluated. The results generated by software Action for the 30-min prediction of one share are shown in Figure 1. The top tables represent (i) the fitting phase of the coefficients (μ , σ) for the initial 60% of the data in that time interval and (ii) GBM-based price predictions (testing) for the remaining 40% of the data.

The graphical analysis is divided into (i) visualization of the probability paper (PP) graph for the conducted AD test, in which the adequacy of the analyzed statistical model to the data is considered useful in cases in which the distribution of the points is a straight line and (ii) the QQPlot graph, which considers that the two analyzed probability distributions (actual versus Gaussian) are similar in cases in which the points lie on a straight line.

The table with the *p-value* of the performed AD test is presented. The obtained *p-value* was 0.6895, indicating that, at a significance level of 5%, the hypothesis that the residuals obtained by GBM projection follow a normal distribution is accepted (Figure 1) The QQPlots of three other stocks (PCAR4, GOAU4, LIGT3) derived from the tests in the 30-min time interval are shown in Figure 2. The first two graphs are examples of approval, and the last graph is an example of rejection of hypothesis H1.

As a strategy to organize and facilitate the interpretation of the results, Tables 2 and 3 show the obtained results in the two capital markets (B3 and NYSE). The average success rate in the 30-min time interval was 87.0% and 70.8% for B3 and NYSE, respectively. The mean rate of success in the 60-min interval was 70.8% and 62.6% for B3 and NYSE, respectively. The overall mean rate of success was 78.8% (30 min) and 66.6% (60 min).

The rate of success of adherence to normal distribution in the 60 and 30-min time intervals was compared for determining the best time interval for the GBM-based forecast. H2 was validated at a significance level of 1%, i.e., the mean success rate in the normality test in the 30-min interval was higher than that in the 60-min interval in B3. However, the same result was not observed in the NYSE.



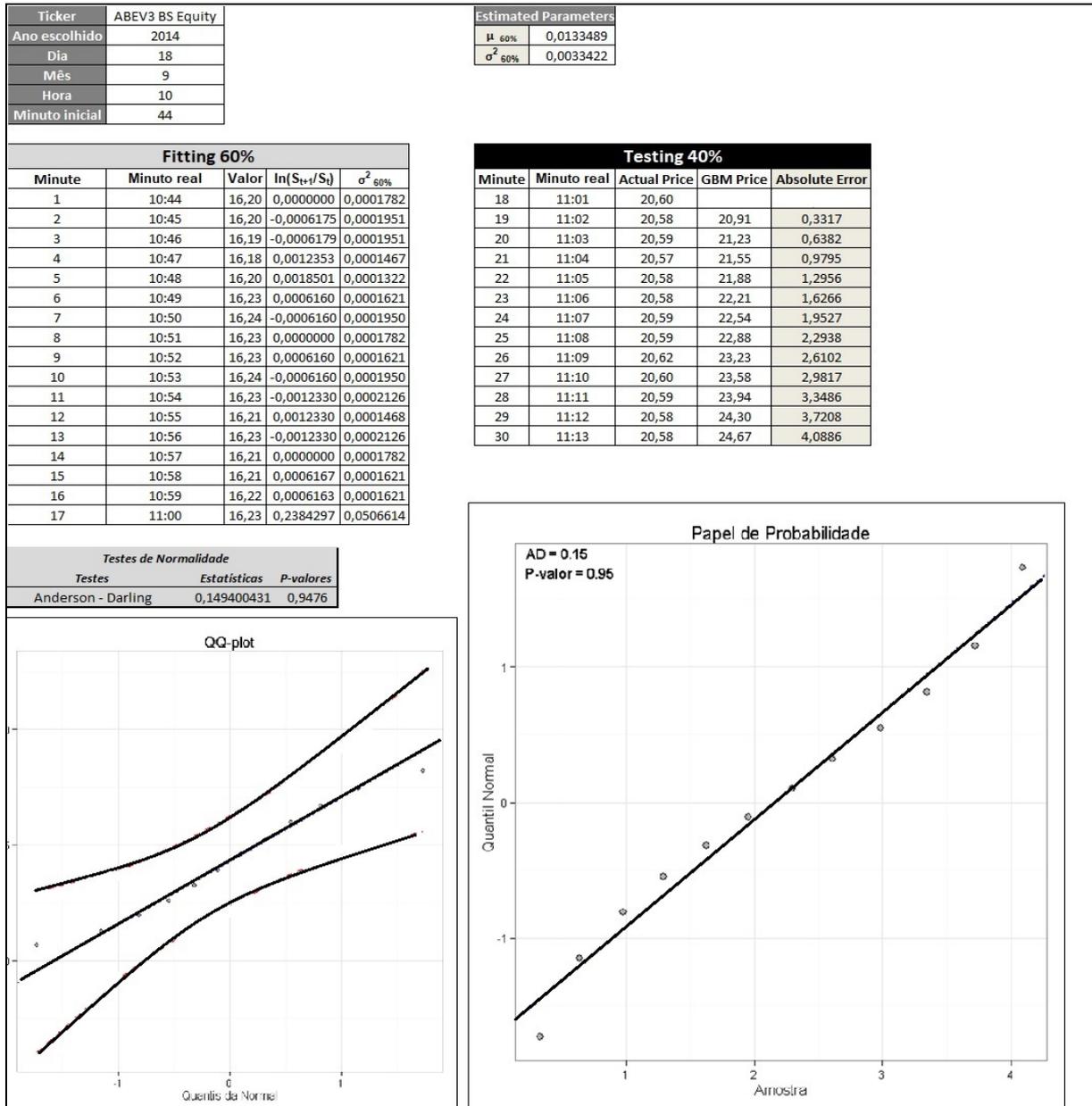


Figure 1: GBM-based Prediction and AD Test in the 30-min time interval.

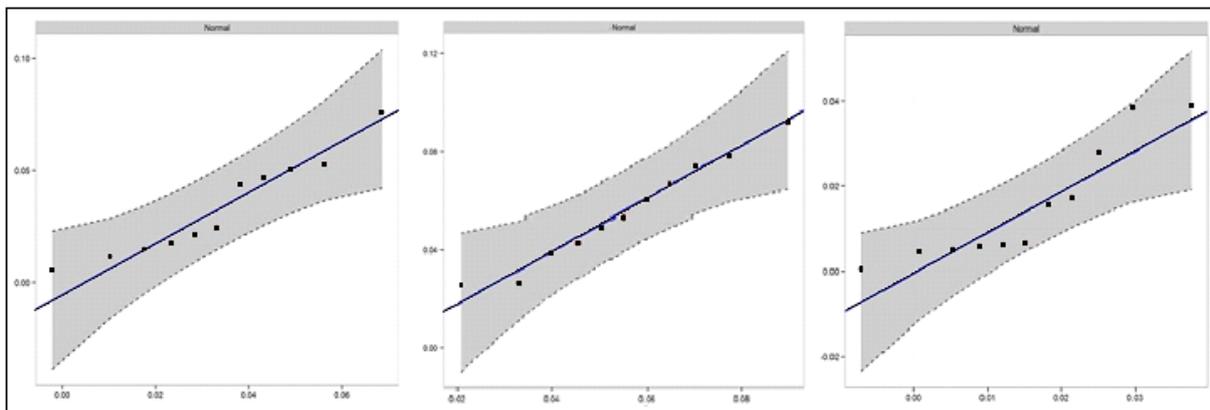


Figure 2: QQPlots (PCAR4, GOAU4, LIGT3)

Table 2: Success rate of the normality test for B3 stocks in 60-min and 30-min time intervals

Date	Stock	Success rate (60 min)	Success rate (30 min)
2014/09/18	BBAS3	66.7	92.3
2015/03/06	PDGR3	100.0	100.0
2014/09/19	GOAU3	100.0	100.0
2014/09/22	TIMP3	50.0	84.6
2015/01/20	BNBR3	20.0	66.7
2014/12/04	USIM5	33.3	69.2
2015/03/05	JBSS3	33.3	61.5
2015/03/18	PCAR4	33.3	100.0
2014/11/13	BBDC4	66.7	84.6
2014/11/07	PFRM3	100.0	100.0
2015/02/05	LIGT3	33.3	53.9
2015/01/12	RHDS3	100.0	100.0
2014/10/27	GOAU4	50.0	76.9
2014/10/15	JBDU3	100.0	100.0
2015/02/20	LIXC4	100.0	100.0
2015/01/06	CYRE3	83.3	84.6
2015/01/23	CESP6	40.0	66.7
2014/10/09	CRUZ3	83.3	84.6
2014/12/03	ABEV3	83.3	69.2
2014/12/10	EMBR3	50.0	92.3
2014/10/29	DASA3	100.0	100.0
2014/11/19	TUPY3	100.0	100.0
2015/01/30	ELET3	100.0	100.0
2014/11/05	HGTX3	33.3	76.9
2015/03/11	VAGR3	100.0	100.0
2014/10/14	BRML3	66.7	84.6
2015/01/09	VISA34	80.0	100.0
2014/10/08	KROT3	50.0	69.2
2014/10/06	BRKM5	66.7	92.3
2015/03/30	BOBR4	100.0	100.0
Average		70.8	87.0

Table 3: Success rate of the normality test for NYSE stocks in 60-min and 30-min time

Date	Stock	Success rate (60 min)	Success rate (30 min)
2015/03/06	RAX UN	83.3	72.7
2014/10/30	PHM UN	50.0	83.3
2015/02/11	CLX UN	50.0	87.5
2014/10/29	PCP UN	100.0	75.0
2015/03/03	ABB UN	50.0	60.0
2015/04/02	SLB UN	33.3	66.7
2015/02/20	DAL UN	66.7	75.0
2015/03/30	SUNE UN	50.0	75.0
2015/02/12	PX UN	66.7	90.0
2014/11/10	UA UN	66.7	81.8
2014/10/31	SWFT UN	83.3	81.8
2015/02/23	RCL UN	50.0	37.5
2014/11/11	EMN UN	66.7	88.9
2015/01/27	OIS UN	0.0	40.0
2014/09/30	PAY UN	33.3	60.0
2015/01/20	BURL UN	83.3	66.7
2014/10/07	DD UN	33.3	80.0
2014/11/20	GIS UN	100.0	33.3

2014/09/30	RAI UN	66.7	0.0
2015/01/15	OFC UN	80.0	80.0
2014/12/30	FRO UN	100.0	100.0
2015/01/05	HME UN	33.3	100.0
2014/12/29	KIM UN	80.0	100.0
2015/02/26	HP UN	16.7	58.3
2015/03/10	UHS UN	33.3	0.0
2015/01/02	BBL UN	83.3	66.7
2015/01/13	TNK UN	100.0	100.0
2014/10/13	PH UN	83.3	88.9
2014/12/09	LHO UN	80.0	88.9
2014/12/18	CE UN	100.0	100.0
2014/11/24	FLR UN	16.7	58.3
Average		62.6	70.8

4.2. Assessment of the Profitability of the Portfolios using GBM-based Prediction

The decisive test for assessing the efficiency of the GBM model is its ability to form profitable portfolios. Because of the restrictions explained in the methodology, it was possible to form 85 distinct minimum-risk portfolios with a goal of achieving a positive rate of return (≥ 0). A minimum positive rate of return was chosen because the optimization model could freely identify all possible profitable portfolios. The configuration of the MV model used in the test was as follows:

$$\text{Min}Z = \sum_{i=1}^N \sum_{j=1}^N x_i x_j \sigma_{ij} \quad (7)$$

Subject to:

$$\sum_{i=1}^N x_i \mu_i \geq 0 \quad (8)$$

$$\sum_{i=1}^N x_i = 1 \quad (9)$$

$$x_i \geq 0 \quad i = 1, \dots, N \quad (10)$$

A summary of the statistical procedures and the rates of returns of the formed portfolios are shown in Tables 4 and 5, respectively. In addition to the stocks shown in Tables 2 and 3, the following stocks were evaluated: ACCO UN, AMG UN, APD UN, ARES UN, FN UN, GLOB UN, HQL UN, IGT UN, LAD UN, MXF UN, MY UN, OMAM UN, SGF UN, WES UM.

Table 4: Statistical summary

Statistical procedure	Objective	Level of significance	Result
Minimum sample size	Determine the minimum sample size for the AD tests	5%	$n_{30}=n_{60}=171$ $n_{total} = 684$
Non-parametric AD test (H1)	Evaluate the normality of GBM residuals	5%	Adherence to normal distribution in 78.8% of cases for $n = 633$ (30-min interval) and 66.6% for $n = 372$ (60-min interval)
Hypothesis test (H2)	Comparison of the normality rate in 30-min and 60-min intervals on the B3	1%	There was evidence of $\mu_{30\ min} > \mu_{60\ min}$
Hypothesis test (H2)	Comparison of the normality rate in 30-min and 60-min intervals on the NYSE	1%	There was no evidence of $\mu_{30\ min} > \mu_{60\ min}$
Hypothesis test (H3)	Evaluation of the profitability of the portfolios formed on the B3	1%	There was evidence of $\mu \geq 0$ (profit)
Hypothesis test (H3)	Evaluation of the profitability of the portfolios formed on the NYSE	1%	There was evidence of $\mu \geq 0$ (profit)

Table 5: Rate of return of the formed portfolios (%)

Date	Day	Time	B3 Rate of return	NYSE Rate of return
09/18/2014	Thursday	12:00	0.0410	0.0503
09/23/2014	Tuesday	15:30	0.0146	0.0309
09/25/2014	Thursday	14:00	-0.0357	0.1139
09/30/2014	Tuesday	11:00	0.0168	0.0760
10/01/2014	Wednesday	13:30	-0.0002	-0.0127
10/07/2014	Tuesday	11:30	0.0387	0.0782
10/09/2014	Thursday	14:30	-0.2585	-0.0008
10/14/2014	Tuesday	12:00	0.0332	-0.0456
10/16/2014	Thursday	10:30	0.2418	-0.0195
10/21/2014	Tuesday	14:30	0.1463	0.0682
10/23/2014	Thursday	09:30	0.0523	-0.0469
10/29/2014	Wednesday	10:00	-0.0007	0.0494
11/05/2014	Wednesday	15:00	0.0559	0.0083
11/11/2014	Tuesday	11:30	0.0609	0.0346
11/13/2014	Thursday	13:00	0.0659	0.0229
11/20/2014	Thursday	12:30	(Holiday)	-0.0501
11/25/2014	Tuesday	13:00	-0.4736	0.0635
11/27/2014	Thursday	10:30	0.0524	(Holiday)
12/03/2014	Wednesday	09:30	0.0405	0.3983
12/16/2014	Tuesday	10:00	-0.0570	0.2110
12/18/2014	Thursday	11:30	0.0588	0.1223
12/23/2014	Tuesday	11:00	0.1246	-0.0044
01/06/2015	Tuesday	14:00	0.0931	0.0293
01/08/2015	Thursday	15:30	0.2147	0.0608
01/13/2015	Tuesday	13:00	0.0096	-0.3452
01/15/2015	Thursday	10:00	0.0485	0.1552
01/20/2015	Tuesday	10:30	-0.0207	0.0407
01/27/2015	Tuesday	09:30	0.0884	-0.0252
01/29/2015	Thursday	15:30	0.0741	0.0064

02/03/2015	Tuesday	13:30	0.1565	0.0713
02/05/2015	Thursday	12:00	-0.0148	0.0877
02/11/2015	Wednesday	15:30	0.0376	0.0713
02/17/2015	Tuesday	15:00	(Holiday)	0.0301
02/19/2015	Thursday	09:30	0.0210	0.0891
02/24/2015	Tuesday	11:30	0.4289	0.0101
02/26/2015	Thursday	10:00	0.0080	-0.1313
03/4/2015	Wednesday	13:00	0.1229	0.0469
03/10/2015	Tuesday	11:00	0.0486	0.0330
03/12/2015	Thursday	13:30	0.2579	0.0899
03/18/2015	Wednesday	14:00	0.3486	0.1477
03/24/2015	Tuesday	15:30	0.0717	-0.0030
03/26/2015	Thursday	14:00	-0.2193	-0.0121
03/31/2015	Tuesday	10:30	0.0652	0.0812
04/02/2015	Thursday	14:00	0.2333	0.0537
Annual Return			14.61	10.63
CI ($\alpha=0.05$)			14.61 \pm 4.21	10.63 \pm 3.07

5. DISCUSSION

The results confirmed that GBM and HFT can be used by small investors. The sequence of the performed tests was logical. We first evaluated whether GBM could provide satisfactory results for forecasting prices at 1-min intervals. After that, the best time interval of data accumulation (30- or 60-min) for trading was analyzed. Finally, we assessed whether the portfolios formed by GBM-based price prediction were profitable.

It should be noted that ordinary investors have public access only to the negotiated prices of the stocks in each time interval. The decision structure used in this study considered that small investors had resources and had access only to the negotiated prices of the stocks. Professional investors (e.g., brokers and pension funds) have privileged access to other data (e.g., stock lots traded per price) that allow better price forecasts.

Following the tests, each stage supported the execution and results of the following phases. The results of the applied tests indicated that GBM might be used in decision-making by small investors. Similar to other studies that evaluated different markets and time intervals (ABIDIN; JAFFAR, 2012; REBOREDO ET AL., 2013; REDDY; CLINTON, 2016; ZHOU, 2015), our results confirmed the quality of GBM-based forecasts in the B3 and NYSE (H1 hypothesis test), particularly in the 30-min interval. For a sample size based on the conservative premise of $p^* = 0.50$, the obtained error margins were lower than the margin of 7.5%. As an example, the error margin of the success rate of B3 in the 30-min interval was 5% after sampling.

Decision-making based on data collected in short intervals is one of the foundations of HFT (MENKVELD, 2013). HFT made at short intervals reduces the volatility of trading

returns (CHABOUD ET AL., 2014; HASBROUCK; SAAR, 2013). Therefore, the shorter is the time interval, the lower is the exposure to market volatility and, consequently, the better are the conditions of predicting prices. It is not surprising that this type of negotiation was increased and is ideal for markets that present technological infrastructure capable of offering data in milliseconds, such as the NYSE and Chi-X.

Therefore, the results of the H2 test for the 30-min success rates are consistent, especially for the Brazilian market. There was no significant difference between the 30- and 60-min time intervals probably because of the higher predictability and the lower volatility of the American market, and therefore any of these intervals might be used. Nonetheless, the success rate was higher in the 30-min interval.

One of the assumptions for statistical inference analysis, such as hypothesis testing, is the independence between the evaluated events. An interval of at least two business days was used between the dates and times selected for portfolio formation. This strategy decreased the number of formed portfolios but increased the reliability of the confirmation of profitable portfolios evaluated by hypothesis H3.

The rate of return depends on market characteristics, including market size (e.g., number of participants and traded volume), volatility, liquidity, trading costs, and opportunity costs. Although the mean return of GBM portfolios on B3 was higher than that on the NYSE and is consistent with the observed success rates, these results should be viewed individually.

The annual rates of return (250 business days) were high even when discounting the inflation of the respective periods and markets. In the Brazilian capital market, the estimated mean annual rate of return was 14.61% for an inflation of 11.09% with an actual gain of 3.17%. In the United States, the actual gain of 9.81% can be considered exceptional and was obtained from an estimated mean annual return of 10.63% for an inflation of 0.75%.

6. CONCLUSIONS

This study is the first to demonstrate that HFT characteristics (intraday trading, short time forecasts, zero inventory) can be used by small investors for investment allocation. GBM was feasible for predicting stock prices in two capital markets (Brazil and the United States) with different rates of liquidity and volatility.

The adherence of the residuals to normal distribution, evaluated by the AD test, was satisfactory and consistent because the percentage of adherence in 30-min intervals (78.8%) was higher than that in 60-min intervals (66.6%). With respect to the profitability of the



portfolios, the use of the Markowitz mean-variance model indicated higher assertiveness in gaining profit, and therefore the measurements of the risk and yield were satisfactory. High actual rates were obtained by annualizing the mean percentage returns.

GBM was assessed in free 1-min intervals obtained on the B3 and NYSE. A promising line of research is the incorporation of this model into other portfolio optimization models (CVaR, Downside risk) to compare the efficiency of portfolio formation under different risk conditions. Furthermore, this study opens the way for exploring the integration of the risk-return duality into the concept of liquidity because, in HFT negotiations, it is essential to guarantee the sale of formed portfolios.

REFERENCES

- ABENSUR, E. O. (2014) Markov Chain Portfolio Liquidity Optimization Model. **Independent Journal of Management & Production**, v. 5, n. 2, p. 360-380.
- ABIDIN, S. N. Z.; JAFFAR, M. M. (2012) A Review on Geometric Brownian Motion in Forecasting the Share Prices in Bursa Malaysia. **World Applied Sciences Journal**, v. 17, p. 87-93.
- ACTION STAT PRO PARA EXCEL. **ACTION STAT PRO PARA EXCEL**. Estatcamp, Available: <http://www.portalaction.com.br/>.
- AHMADI-JAVID, A. (2012) Entropic Value-at-Risk: A New Coherent Risk Measure. **Journal of Optimization Theory and Applications**, v. 155, n. 3, p. 1105–1123. Available: DOI: 10.1007/s10957-011-9968-2.
- ANDERSON, D. R.; SWEENEY, D. J.; WILLIAMS, T. A. (2010) **Statistics for Business and Economics**, Nashville: South-Western College Pub.
- BLACK, F.; SCHOLES, M. (1973) The Pricing of Options and Corporate Liabilities. **Journal of Political Economy**, v. 81, n. 3, p. 637–654.
- BODINEAU, T.; GALLAGHER, I; SAINT-RAYMOND, L. (2016) The Brownian Motion as the Limit of a Deterministic System of Hard-Spheres. **Inventiones Mathematicae**, v. 203, n. 2, p. 493-553.
- CARRADORI, R. G.; RAMOS, P. S. (2014) Avaliação de Testes de Normalidade Implementados no Programa R por Simulação de Monte Carlo. **Revista da Estatística UFOP**, v. 3, n. 2, p. 33-41.
- CASTELLANO, R.; CERQUETI, R. (2014) Mean–Variance portfolio selection in presence of infrequently traded stocks. **European Journal of Operational Research**, v. 234, n. 2, p. 442-449. Available: <https://doi.org/10.1016/j.ejor.2013.04.024>.
- CHABOUD, A. P. ; CHIQUOINE, B.; HJALMARSSON, E.; VEGA, C. (2014) Rise of the Machines: Algorithmic Trading in the Foreign Exchange Market. **The Journal of Finance**, v. 69, n. 5, p. 2045-2084. DOI: 10.1111/jofi.12186.

- CHOI, T. M.; CHUI, C. H. (2012) Mean-Downside-Risk and Mean-Variance Newsvendor Models: Implications for Sustainable Fashion Retailing. **International Journal of Production Economics**, v. 135, n. 2, p. 552-560. DOI: <https://doi.org/10.1016/j.ijpe.2010.10.004>.
- COLMAN, D. L.; WIENANDTS, M. E. L.; DE PIETRO, T. C. (2013). Análise de dados intraday usando a teoria da matriz aleatória. **Academia**. Available: http://www.academia.edu/6275322/2013_-_An%C3%A1lise_de_dados_intraday_usando_a_teor%C3%ADa_da_matriz_aleat%C3%B3ria.
- CUI, X.; GAO, J.; LI, X.; LI, D. (2014) Optimal Multi-Period Mean-Variance Policy Under No-Shorting Constraint. **European Journal of Operational Research**, v. 234, n. 2, p. 459-468. DOI: <https://doi.org/10.1016/j.ejor.2013.02.040>.
- HASBROUCK, J.; SAAR, D. S. (2013) Low-latency Trading. **Journal of Financial Markets**, v. 16, n. 4, p. 646-679. DOI: <https://doi.org/10.1016/j.finmar.2013.05.003>.
- HENDERSHOTT, T.; JONES, C. M.; MENKELVELD, A. J. (2011) Does Algorithmic Trading Improve Liquidity? **The Journal of Finance**, v. 66, n. 1, p. 1-33. DOI: [10.1111/j.1540-6261.2010.01624.x](https://doi.org/10.1111/j.1540-6261.2010.01624.x).
- HILLIER, F. S.; LIEBERMAN, G. J. (2015) **Introduction to Operations Research**, New York: McGraw-Hill.
- IMAN, R. L.; CONOVER, W. J. (1983) **A Modern Approach to Statistics**. New York, John Wiley & Sons.
- ITO, K. (1944) Stochastic Integral. In: Imperial Academy, 20, Tokyo, **Proceedings.....**, Tokyo: Imperial Academy. Available: https://projecteuclid.org/download/pdf_1/euclid.pja/1195572786.
- IWAKI, H.; LUO, L. (2013) An Empirical Study of Option Prices under the Hybrid Brownian Motion Model. **Journal of Mathematical Finance**, v. 3, p. 329-334. Available: <http://dx.doi.org/10.4236/jmf.2013.32033>.
- KEIM, D.; STAMBAUGH, R. (1984) A Further Investigation of the Weekend Effect in Stock Returns. **The Journal of Finance**, v. 39, n. 3, p. 819-835. DOI: [10.1111/j.1540-6261.1984.tb03675.x](https://doi.org/10.1111/j.1540-6261.1984.tb03675.x).
- KHARROUBI, I.; LIM, T. ; NGOUPEYOU, A. (2013) Mean-Variance Hedging on Uncertain Time Horizon in a Market with a Jump. **Applied Mathematics & Optimization**, v. 68, n. 3, p. 413-444. DOI: <https://doi.org/10.1007/s00245-013-9213-5>.
- KOGAN, L.; PAPANIKOLAOU, D. (2014) Growth Opportunities, Technology Shocks, and Asset Prices. **The Journal of Finance**, v. 69, n. 2, p. 675-718. DOI: [10.1111/jofi.12136](https://doi.org/10.1111/jofi.12136).
- LAGE, E. L. D. C. (2011) **Avaliação de Projetos de Shopping Center: Aplicação da Teoria de Opções Reais**. Dissertation (Master in Production Engineering). Available: <http://bibliotecadigital.fgv.br/dspace/handle/10438/6541>.
- LIQUI, A.; PONCET, P. (2016) Understanding Dynamic Mean Variance Asset Allocation. **European Journal of Operational Research**, v. 254, n. 1, p. 320-337. Available: <https://doi.org/10.1016/j.ejor.2016.04.003>.
- MARKOWITZ, H. (1952) Portfolio Selection. **The Journal of Finance**, v. 7, n. 1, p. 77-91. DOI: [10.1111/j.1540-6261.1952.tb01525.x](https://doi.org/10.1111/j.1540-6261.1952.tb01525.x).

- MARKOWITZ, H. (2014) Mean–Variance Approximations to Expected Utility. **European Journal of Operational Research**, v. 234, n. 2, p. 346-355. Available: <https://doi.org/10.1016/j.ejor.2012.08.023>.
- MCNEIL, J. A.; FREY, R.; EMBRECHTS, P. (2015) **Quantitative Risk Management: Concepts, Techniques and Tools**, United Kingdom: Princeton University Press.
- MENKVELD, A. (2013) High Frequency Trading and the New Markets. **Journal of Financial Markets**, v. 16, n. 4, p. 712-740. DOI: doi.org/10.1016/j.finmar.2013.06.006.
- NOYAN, N.; RUDOLF, G. (2013) Optimization with Multivariate Conditional Value-at-Risk Constraints. **Operations Research**, v. 61, n. 4, p. 990-1013. DOI: <https://doi.org/10.1287/opre.2013.1186>.
- PENTAGNA, A. P. (2015) **High Frequency Trading: Riscos e Propostas de Regulamentação**. Monography. Available from: <http://acervodigital.ufpr.br/bitstream/handle/1884/42903/MONOGRAFIA05-2015.pdf?sequence=1&isAllowed=y>.
- QIN, Z. (2015) Mean-Variance Model for Portfolio Optimization Problem in the Simultaneous Presence of Random and Uncertain Returns. **European Journal of Operational Research**, v. 245, n. 2, p. 480-488. DOI: <https://doi.org/10.1016/j.ejor.2015.03.017>.
- REDDY, K.; CLINTON, V. (2016) Simulating Stock Prices Using Geometric Brownian Motion: Evidence from Australian Companies. **Australian Accounting, Business and Financial Journal**, v. 10, n. 3, p. 23-47. DOI:10.14453/aabfj.v10i3.3.
- REBOREDO, J. C.; RIVERA-CASTRO, M. A.; MIRANDA, J. G. V.; GARCIA-RUBIO, R. (2013) How fast do stock prices adjust to market efficiency? Evidence from a detrended fluctuation analysis. **Physica A**, v. 392, p. 1631-1637. DOI:10.1016/j.physa.2012.
- ROCKAFELLAR, R.; URYASEV, S. (2000) Optimization of Conditional Value-at-Risk. **Journal of Risk**, v. 3, n. 2, p. 21-41.
- ROSS, S. **Introduction to Probability Models** (2014), San Diego: Elsevier.
- SECURITIES AND EXCHANGE COMMISSION-SEC. (2014) **Equity Market Structure Literature Review Part II: High Frequency Trading**. Available: https://www.sec.gov/marketstructure/research/hft_lit_review_march_2014.pdf
- SEYF, H. R.; NIKAAEIN, B. (2012) Analysis of Brownian Motion and Particle Size Effects on the Thermal Behavior and Cooling Performance of Microchannel Heat Sinks. **International Journal of Thermal Sciences**, v. 58, n. 1, p. 36-44. DOI: <https://doi.org/10.1016/j.ijthermalsci.2012.02.022>.
- SHENG, S. P.; LIU, M.; SAIGAL, R. (2015) Data-Driven Channel Modeling Using Spectrum Measurement. **IEEE Transactions on Mobile Computing**, v. 14, n. 9, p. 1794-1805. DOI: 10.1109/TMC.2014.2374152.
- SHIN, H.; JUNG, Y.; JEONG, C.; HEO, J. H. (2012) Assessment of Modified Anderson–Darling Test Statistics for the Generalized Extreme Value and Generalized Logistic Distributions. **Stochastic Environmental Research and Risk Assessment**, v. 26, n. 1, p. 105-114.
- SIGMAN, K. (2006) **Notes on Financial Engineering** (Columbia University). Available: <http://www.columbia.edu/~ks20/FE-Notes/4700-07-Notes-GBM.pdf>.

WIENER, N. (1923) Differential Space, **Journal of Mathematical Physics**, v. 2, p. 131–174
DOI: 10.1002/sapm192321131.

ZHANG, S.; ZHOU, W. (2015). Probabilistic Characterization of Metal-Loss Corrosion Growth on Underground Pipelines Based on Geometric Brownian Motion Process. **Structure and Infrastructure Engineering**, v. 11, n. 2, p. 238-252. DOI:
<http://dx.doi.org/10.1080/15732479.2013.875045>.

ZHOU, Q. X. (2015) The Application of Fractional Brownian Motion in Option Pricing. **Structure and Infrastructure Engineering**, v. 10, n. 1, p. 173-182.